Four–manifolds and knot concordance.

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Topology deals with global or large-scale structures of objects. Geometric topology is the study of manifolds and its ultimate goal is the classification of manifolds within a certain framework (topological, piecewise linear, smooth, simply-connected, symplectic, etc.). Since classification is in many cases elusive (and at times impossible [21, Section 1.2]), topologists develop analytic and algebraic invariants that detect the differences between manifolds.

I specialize in 4–manifolds and knot concordance. Dimension four is special, as it is the only dimension in which a fixed homeomorphism type of closed manifolds can be represented by infinitely many diffeomorphism types, and the only dimension in which there exist exotic $\mathbb{R}^n$'s (manifolds homeomorphic but not diffeomorphic to $\mathbb{R}^n$). Thus, 4–manifold theory has a distinct character and topological and smooth 4–manifolds require radically different approaches. The 1980's saw the development of new and more complex ways of investigating the different approaches necessary to study topological and smooth 4–manifolds. Freedman's extraordinary work on topological embeddings of 2–dimensional disks in 4-manifolds [12, 13, 14] extended surgery theory to topological 4–manifolds and resulted in a classification theorem for topological 4–manifolds. Donaldson's application of the instanton solutions of the Yang-Mills equations as a tool in 4-manifold theory [5, 6, 7] showed that in some cases smooth surgery is impossible even though topological surgery can be accomplished, and it additionally provided strong obstructions to the existence of smooth structures of topological 4–manifolds. As for knots, embeddings of 2–disks into the 4–ball can be used to define the equivalence relation on the set of knots in S^3 known as concordance. Two knots K and J are concordant if $K\# -J$ bounds a 2–disk in the 4–ball, and depending on whether the disks are topological or smooth, we get analogous notions of concordance. As a result of Freedman's and Donaldson's work, there exist many locally flat topological embeddings of 2–disks in 4–manifolds that cannot be smoothed. Knots that bound this type of disks give rise to exotic $\mathbb{R}^4$'s and their existence is evidence that knot concordance lies at the center of interesting interactions between all low dimensions.

My main research interests are knot concordance, homology cobordism groups, gauge theory, and trisections. The following is an outline of the sections of this document, together with a brief summary of my past, current, and future research projects. More details are included in subsequent pages.

1. **Knot Concordance and Satellites:** In [36, 37] I use the theory of $SO(3)$ instantons to establish an obstruction for a family of iterated Whitehead-like satellites of positive torus knots to be independent in smooth concordance, while trivial in topological concordance. In joint work with Matt Hedden [24] we provide a general criterion for the image of a satellite operation to generate an infinite rank subgroup of the smooth concordance group. Our criterion can be checked entirely in terms of Seifert surfaces, and applies widely, most notably to patterns for which the corresponding operators on the topological concordance group are zero. In a future project I will study of concordance classes of links $P \sqcup \gamma$ to find sufficient conditions for a satellite operation of zero winding number to have infinite rank. Additionally, in a current project with Allison Miller we develop criteria to obstruct a pattern P from giving a homomorphism in either $\mathcal{C}_{\text{TOP}}$ or $\mathcal{C}$.

2. **Floer Homologies:** Donaldson's work on instantons can be extended to produce homology theories that give strong invariants of knots and 3–manifolds. With Yoshihiro Fukumoto and Paul Kirk [16] we studied the chain complex that gives rise to the instanton knot homology of Kronheimer and Mrowka, focusing our attention on torus and pretzel knots. We provided a method for identifying the generators of their instanton homology, completely determined the subspace of binary dihedral representations, and identified their traceless representations. In a future project I will use the tech-

niques developed in [16] and work of Herald and Kirk-Klassen, to compute Chern-Simons invariants of branched covers of homology spheres. Next, an ongoing project with Tye Lidman and Christopher Scaduto is aimed at obtaining computations of Froyshov's instanton invariant for Seifert fibered spheres and homology spheres arising from surgery along positive knots. Lastly, an ongoing project with Tye Lidman extends Baldwin and Sivek's result on the existence of $SU(2)$ representations to the include all 3–manifolds that arise as the boundary of Stein manifolds with the same homology as the 4–ball.

3. **Trisections of Four-manifolds:** A 3–dimensional counterpart to concordance is obtained by studying 3–manifolds as the boundary of 4–manifolds. In joint work with David Gay and Nick Castro [2, 3], we show that 4–manifolds with boundary can be completely described in terms of diagrams of curves in surfaces. Furthermore, we describe how to obtain such diagrams from a handle decomposition of a 4–manifold with boundary, and an open book decomposition of the boundary. A future project explores the connection between the topology of the trisection and the geometry of the contact structure by establishing a relationship between trisection diagrams and Ozvath and Szabo's contact invariant. Finally, the AIM SQuaRE project "Trisections, Knotted Surfaces, and Symplectic 4-Manifolds", joint with P. Lambert-Cole, J. Meier, P. Melvin and L. Starkston, focuses on the study of trisections and bridge trisections and hopes to find connections to symplectic and complex geometry.

1. Knot Concordance and Satellites

A knot is a smooth embedding of S^1 into S^3. Two knots K_0 and K_1 are said to be smoothly concordant if there is a smooth embedding of $S^1 \times [0,1]$ into $S^3 \times [0,1]$ that restricts to the given knots at each end. Requiring such an embedding to be locally flat instead of smooth gives rise to the weaker notion of topological concordance. Both kinds of concordance are equivalence relations, and the sets of smooth and topological concordance classes of knots, denoted by $\mathcal{C}$ and $\mathcal{C}_{\text{TOP}}$ respectively, are abelian groups with connected sum as their binary operation. In both cases the identity element is the concordance class of the unknot and the knots in that class are know respectively as smoothly slice and topologically slice. There are profound differences between these concordance groups and although much progress has been made to understand these differences, many questions remain unanswered. For example, identifying the set of knots that are topologically slice but not smoothly slice is a challenging topic, among other reasons because these knots reveal subtle properties of differentiable structures in dimension four [21, p. 522]. In addition, it is still unknown whether the groups possess elements of any finite order other than two.

One powerful approach for analyzing concordance groups and identifying the set of knots that are topologically slice but not smoothly slice comes from satellite operations. To define these, start with a given knot P embedded in an unknotted solid torus $V \subseteq S^3$. Assign to an arbitrary knot $K \subseteq S^3$ the image of P under the embedding of V in S^3 that knots V as a tubular neighborhood of K. This is the satellite knot with pattern $P \subseteq V$ and companion K, and is denoted by $P(K)$. Since framings of a properly embedded annulus in $[0,1] \times S^3$ are naturally identified with framings of either circle on the boundary, it follows that the satellite operator P descends to concordance classes. Thus, for any knot $P \subset V$ we obtain a self-map on both $\mathcal{C}$ and $\mathcal{C}_{\text{TOP}}$. In previous and ongoing research projects I have explored the difference of satellite operations in the smooth and topologically flat category, answered questions about their rank, and investigated whether any such map is a homomorphism.

1.1. Satellites of Infinite Rank

Questions about the injectivity and surjectivity of functions are pervasive in mathematics, and in the case of finitely generated abelian groups and homomorphisms, answers can be obtained using ranks. Although this is not true for more general abelian groups, exploring ranks in the more general setting is still useful. As it turns out, it is very hard to determine the injectivity or surjectivity of satellite operators and so in [24, 36, 37] I instead investigated their rank. In this case we say that $P : \mathcal{C} \to \mathcal{C}$ has infinite rank if there exists an infinite linearly independent subset $\mathcal{R}$ of $\mathcal{C}$ such that $P(\mathcal{R})$ is also infinite and linearly independent.

Whitehead doubles are a notable example of satellite operations and are obtained by using the Whitehead link as the pattern of the operation. Similar examples arise by considering Whitehead-like patterns D_n and their iterates D_n^r. Because the knot D_n^r is unknotted in S^3 and trivial in $H_1(V;\mathbb{Z})$, every satellite knot with pattern D_n^r is topologically slice and so the function $D_n^r : \mathcal{C}_{\text{TOP}} \to \mathcal{C}_{\text{TOP}}$ is constant. Therefore, classical invariants do not detect information about their smooth concordance type and smooth techniques like gauge theory are necessary. In [36, 37] I established an obstruction for a family of iterated Whitehead-like satellites of positive torus knots to be dependent in the smooth concordance group, thus extending a theorem of Hedden-Kirk [23]. I introduced definite cobordisms from the branched covers of the iterated doubles to Seifert fibered homology 3-spheres which allowed me to sidestep the complicated computations of [23]. The precise statement of my result is the following.

Theorem (PC. [36, 37])**.** *Let $\{(p_i, q_i)\}_i$ be a sequence of relatively prime positive integers and $\{n_i\}_i$, $\{r_i\}_i$ sequences of positive and even integers. If the sequences satisfy*

$$p_i q_i(2n_i p_i q_i - 1) < p_{i+1}q_{i+1}(n_{i+1}p_{i+1}q_{i+1} - 1),$$

then the collection $\left\{D_{n_i}^{r_i}\left(T_{p_i q_i}\right)\right\}_{i=1}^{\infty}$ is an independent family in the smooth concordance group.

For a satellite operator $P : \mathcal{C} \to \mathcal{C}$, define its winding number w as the algebraic intersection between P and a meridional disk for the solid torus V. If a pattern has non-zero winding number, Levine-Tristram signatures show that there is an infinite family of independent torus knots whose image under P is also independent. The winding number zero case is significantly harder since signatures are independent of the choice of K, and in a recent article with Matt Hedden [24] we provide a general criterion to guarantee that a satellite operator with winding number zero has infinite rank. We prove our theorem in the context of $SO(3)$ gauge theory, using instanton moduli spaces of adapted bundles over 4–manifolds in conjunction with the Chern-Simons invariants of flat connections on the 3–manifolds arising as cross sections of their ends. The precise statement of our result is the following:

Theorem (Hedden-PC. [24])**.** *Let $P \subset S^1 \times D^2$ be a pattern with winding number zero, and let J denote one of the lifts of ∂D^2 to the branched double cover $\Sigma(P(U))$. Let l be the rational linking number of J in $\Sigma(P(U))$, taken with respect to the lift of the natural framing of ∂D^2. If l is non-zero, then P has infinite rank in $\mathcal{C}$.*

This result is a vast generalization of the main theorem of [23] and of [36], and it applies to more general examples of satellite operations. It is exciting in no small part because the condition regarding the rational linking number of J in $\Sigma(P(U))$ can be checked entirely in terms of Seifert surfaces. However, the case of vanishing rational linking number remains open. The operators considered in [37] have vanishing rational linking numbers but have infinite rank nevertheless, so the condition $l \neq 0$ is certainly not necessary (although sufficient). So, in a future project I will investigate the relationship between concordance classes of links $P \sqcup \gamma$ with $\gamma = \partial D^2$, and their rational linking number and is aimed at finding sufficient conditions for a satellite operation of zero winding number to have infinite rank.

1.2. The Homomorphism Problem for Satellites

Although all satellite operators with a slice pattern induce homomorphisms on algebraic concordance, they typically do not on topological or smooth concordance. Hedden [22] showed that when the pattern P is a torus knot and γ is a meridian of the torus where P lies, then $P : \mathcal{C} \to \mathcal{C}$ is not a homomorphism. In fact, he conjectured that satellites are never homomorphisms (except for some trivial examples). In joint work with Allison Miller we explore the satellite homomorphism question and develop general criteria to obstruct a pattern P from giving a homomorphism in either $\mathcal{C}_{\text{TOP}}$ or $\mathcal{C}$. In the smooth category we show that the hypothesis and the techniques from [24] also apply to rule out certain satellite operations from being homomorphisms on $\mathcal{C}$. Our result is the following:

Theorem (Miller-PC. (In progress))**.** *Let P be a pattern described by an unknot γ in the complement of $P(U)$. Call w the winding number of P and assume $|w| \neq 1$.*

- *Let p be the smallest prime dividing w, and let J denote a lift of γ to the branched p-fold cover $\Sigma(P(U))$. Let l be the $\Sigma(P(U))$-rational linking number of J with respect to the lift of the natural framing of γ. If l is non-zero, then there exists a torus knot $T_{r,s}$ for which $P(T_{r,s})\#P(T_{r,-s})$ is not slice. Therefore, P does not induce a homomorphism on $\mathcal{C}$.*
- *If there exists some prime p dividing the winding number of P such that $|H_1(\Sigma_p(P(U)))| \neq 1$ and the lifts of γ to $\Sigma_p(P(U))$ generate $H_1(\Sigma_p(P(U)))$, then P does not induce a homomorphism on $\mathcal{C}_{TOP}$.*

2. Floer Homologies

The concepts of topological and smooth manifolds coincide in dimensions one, two, and three, and the fundamental group determines the diffeomorphism type of a manifold (up to torsion in dimension three). In dimension four, however, topological manifolds are utterly different from smooth manifolds and even simply connected 4–manifolds can have multiple inequivalent smooth structures. A powerful method to distinguish different smooth structures is Donaldson theory. Donaldson [5, 6, 7] pioneered the application of the instanton solutions of the Yang-Mills equations as a tool in 4-manifold theory, and Floer's [10] inspiring use of ideas from Morse theory to said solutions enriched the study of 3–manifolds with his definition of new invariants of homology 3–spheres. Formally, Floer homology is the infinite-dimensional Morse homology of the Chern-Simons functional on the space of connections on a bundle over a 3–manifold. From its inception, it was related to the study of the anti-self-dual Yang-Mills equations on 4–manifolds, and the Donaldson invariants of a 4–manifold with boundary are elements of the Floer homology of its 3–dimensional boundary. Because the critical points for the Chern-Simons functional correspond to representations of the fundamental group of the base 3–manifold, the instanton theory is closely related to the fundamental group, the most basic invariant of a 3–manifold and a pivotal piece in Thurston's geometric approach to the study of 3–manifolds.

The use of spaces of solutions to geometric PDEs to define invariants of smooth manifolds exploits a deep connection between analysis, topology, and physics and is usually referred to as gauge theory. These techniques have been crucial in the recent achievements in smooth topology. For example, "Property P" states that every 3–manifold obtained as non-trivial surgery on a non-trivial knot is not a homotopy 3–sphere. Kronheimer-Mrowka [32] resolved Property P by developing versions of instanton homology for knots and links in a 3–manifold. Additionally, extensions of Donaldson's techniques have been used to show that the homology cobordism group is infinitely generated, and that there is a large gap between the smooth and topological concordance groups.

Some of my research projects investigate the interplay between gauge theory and low-dimensional topology, as well as the ways in which they inform each other. The following subsections describe projects related to the application of gauge theoretic techniques to the study of topological objects.

2.1. Representation Spaces as Lagrangian Intersections

For a manifold M denote by $\mathcal{R}(M)$ the space of representations of its fundamental group. These spaces appear prominently in low-dimensional topology as exemplified by the Casson invariant of an oriented homology 3–sphere. This invariant λ equals half the signed number of (conjugacy classes of) irreducible representations of its fundamental group and it was originally defined in terms of Lagrangian intersections as follows. Given a Heegaard splitting $Y = M_1 \cup M_2$, with separating surface S, the space $\mathcal{R}(Y)$ can be understood as the Lagrangian intersection $\mathcal{R}(M_1) \cap \mathcal{R}(M_2)$ in $\mathcal{R}(S)$, and thus $\lambda(Y)$ equals the algebraic intersection of $\mathcal{R}(M_1)$ with $\mathcal{R}(M_2)$. As an invariant that comes from a representation space of the fundamental group, the Casson invariant reveals in particular that the fundamental group is non-zero. At the time of its definition, before Perelman's proof of the Poincare conjecture and geometrization, Casson's invariant offered a way of tackling potential counterexamples to the Poincare conjecture.

The elements of representations spaces of 3–manifolds correspond to generators of the instanton complex in several Floer homology theories. Because the former are algebraic in nature, they are usually easier to understand than the latter. Yet, because of the correspondence between their elements, it is possible to use $\mathcal{R}(M)$ to define invariants of manifolds. Moreover, when the space of representations of the separating manifold is the pillowcase (a 2–sphere with four corners), computations are much simpler. The following projects emulate Casson's construction of interesting invariants via computations of representation spaces in terms of Lagrangian intersections.

Instanton knot homology: Kronheimer and Mrowka [31, 33, 34] proved remarkable results in knot theory by developing versions of instanton homology for knots and links in a 3–manifold. For Y a 3-manifold and $K \subseteq Y$ a knot, they defined the singular instanton knot homology $I^\natural(Y, K)$ as the homology of a chain complex whose groups are generated by traceless elements of $\mathcal{R}(Y \setminus K)$. While powerful and sophisticated, the constructions of Kronheimer-Mrowka are very intricate from a computational perspective. With Y. Fukumoto and P. Kirk [16] we analyzed the problem of identifying the restriction of the map $\mathcal{R}(B_0 \setminus T_0) \to \mathcal{R}(S^2 \setminus \{a, b, c, d\})$ to the traceless elements. We focused on torus knots and pretzel knots, and hence provided a method for identifying the generators of their instanton homology. Using results from [25] and the fact that the determinant $|\Delta_K(-1)|$ gives a lower bound for the rank of $I^\natural(K)$, we were able to recover the instanton knot homology for $T_{3,7}$, $T_{5,7}$, $T_{5,12}$, and $T_{5,17}$. In addition, for the $(-2, 3, n)$ Pretzel knots we found that the rank of $I^\natural(K(-2, 3, n))$ is given by $n - 2 + 4\left(\left[\!\left[\frac{5n+6}{12}\right]\!\right] - \left[\!\left[\frac{n+6}{12}\right]\!\right]\right)$ (here $[\![q]\!]$ denotes the greatest integer less than q).

Chern-Simons of Double Branched Covers: Chern-Simons (CS) invariants are important invariants of closed 3–manifolds and can be used to obtain information about the homology cobordism class of homology 3–spheres. These invariants were a main ingredient in Fintushel and Stern [9] and Furuta's [17] proof that the homology cobordism group is infinitely generated. In [24, 36, 37] I used bounds for the minimum Chern-Simons invariants to obtain the necessary obstructions for the sliceness of satellite knots. Thus, precise computations of CS will lead to new examples of independent families of knots in smooth concordance. However, computations are scarce even for simple constructions like 2-fold branched covers. One of the few known methods is Kirk and Klassen's formula [29, 30] to compute CS for a 3–manifold decomposed into simpler pieces each of which has torus boundary. They show that CS can be computed from the images of the representation spaces of the pieces into the pillowcase. Additionally, the methods introduced by X.-S. Lin [35] and Herald [26] to define the Casson-Lin invariant for knots in general 3–manifolds can be used to describe the image of $\mathcal{R}(Y \setminus N(K))$ into the pillowcase. I plan to combine these gauge-theoretical methods with algebraic methods like the Redeimeister-Schreier algorithm to produce new computations of Chern-Simons invariants of branched covers of homology spheres.

2.2. Comparing Instanton and Heegaard Floer homologies

A consistent theory of instanton complexes for all 3–manifolds requires a negligible interaction between the irreducible and reducible connections. Floer's admissible bundles and the corresponding abelian groups $I_*(Y,P)$ from [10, 11], Kronheimer and Mrowka's framed instanton homology $I_*^{\#}(Y) = I^*(Y\#T^3, P_0\#Q)$ from [33], and Froyshov's reduced groups $\widehat{I}_*(Y)$ from [15] are three different ways of resolving the technical issues that the reducibles pose. This latter groups $\widehat{I}_*(Y)$ are obtained from $I_*(Y)$ by considering interactions with the trivial connection. Their difference is measured by the Froyshov invariant $h(Y) = \frac{1}{2}\left(\chi(I_*(Y)) - \chi(\widehat{I}_*(Y))\right)$, which induces a surjective homomorphism $h : \Theta^3_{\mathbb{Z}} \to \mathbb{Z}$. The Froyshov invariant can be used to show that the group $\Theta^3_{\mathbb{Z}}$ has a $\mathbb{Z}$ summand, to compute bounds for the 4–ball genus of knots, and by work of Daemi [4] to get even more independent families in $\Theta^3_{\mathbb{Z}}$. In a project with Tye Lidman and Christopher Scaduto we aim to compute h for Seifert fibered spheres and homology spheres arising from surgery along positive knots. To overcome the difficulty posed by the fact that instanton groups are very hard to compute, our approach involves computations of h that do not require perfect knowledge of $\widehat{I}_*(Y)$. Rather than compute reduced instanton groups, we explore the relationship between instanton homology and the hat version of Heegaard Floer homology. We are currently able to show that if K is a knot in S^3 with a positive instanton L-space surgery (for example torus knots), the dimension of $I^{\#}(S^3_n(K))$ is equal to the dimension of $\widehat{HF}(S^3_n(K))$.

Gradings of Floer homologies have been extensively used to define invariants of rational homology 3–spheres and are usually respected by cobordism maps. The correction term $d(Y,t)$ from Heegaard Floer is a noteworthy example of this type of invariants, as is the grading of Ozváth and Szabó's contact invariant. In our project we define a $\mathbb{Z}/2$ grading for Baldwin-Sivek's instanton contact invariant and we relate it to the grading of Ozváth and Szabó's contact invariant via a quantity reminiscent of Gompf's d_3 invariant. We expect to obtain a statement relating the absolute $\mathbb{Z}/2$-gradings on $\widehat{HF}$ and $I^{\#}$ that will allow us to compute h for Seifert fibered spheres that arise as surgery along links of at most two components.

2.3. Instantons and Mazur

Much of what is known about 3–manifolds comes from the knowledge of their fundamental groups and the representations of these groups in Lie groups. Of particular interest are integer homology spheres since their fundamental groups have trivial abelianization. Baldwin and Sivek show in [1] that if a 3–manifold bounds a Stein domain that is not an integer homology ball then its fundamental group admits a nontrivial homomorphism to $SU(2)$. In a project with Tye Lidman we are trying to extend Baldwin and Sivek's result to include all Stein fillable homology spheres (other than S^3), including those that arise as the boundary of an integer homology ball. With that in mind we prove the following:

Theorem (Lidman-PC. (In progress))**.** *Let Y be the boundary of a Mazur manifold W. If $Y \neq S^3$, then $I_*(Y) \neq 0$ and so $\pi_1(Y)$ admits a non-trivial $SU(2)$-representation.*

To achieve our goal we explore the general case of 3–manifolds that bound Stein domains with the same homology of the 4–ball. We are able to prove that if W is is not the 4–ball but admits a handle decomposition with a unique 0-handle and the same number of 1- and 2-handles, then either ∂W admits an $SU(2)$ representation, or the boundary of the 4–manifold obtained after changing the framing of one of the 2-handles admits an $SU(2)$ representation. In light of Baldwin and Sivek's result, we investigate the relationship between the framing change and Stein structures and we expect the maximum Thurston-Bennequin framing will give us a bound on the framing that one needs to change to achieve non-triviality.

3. Trisections of Four-manifolds

In its most succinct version, a trisection of a 4–manifold X is a decomposition of X into three copies of $\natural^k S^1 \times B^3$ that intersect pairwise in 3–dimensional handlebodies, and with triple intersection a closed surface (see [18]). If the 4–manifold in question has non-empty boundary, the trisection surface also has boundary and the intersection of each piece $\natural^k S^1 \times B^3$ with ∂X is required to be the product of an interval with a surface with boundary. Relative trisections induce open book decompositions on the bounding 3–manifolds and can thus be regarded as fillings of open book decompositions. In a collaboration with David Gay and Nick Castro [2, 3], we introduced relative trisection diagrams, showed that they uniquely determine trisected 4–manifolds with connected boundary, and described an algorithm for transforming a Kirby diagram into a trisection diagram. The theory of trisections is an exciting theory which enables one to understand 4–manifolds in terms of curves on surfaces. I am currently part of an NSF funded FRG in trisections together with R. Kirby, A. Thompson, M. Tomova, D. Gay, A. Zupan, and J. Meier. This focused research group aims to exploit this new perspective on four–manifolds to solve hard problems in 4–manifold topology that have proved intractable to this day.

3.1. Relative trisections as fillings of open books and the HF Contact Invariant

A relative trisection diagram is a tuple $(\Sigma, \alpha, \beta, \gamma)$, where Σ is a surface with boundary, and α, β and γ are each systems of simple closed curves such that each diagram (Σ, α, β), (Σ, β, γ) and (Σ, γ, α) is handle slide and diffeomorphism equivalent to a certain standard sutured Heegaard diagram. In [2] we show that relative trisection diagrams, besides describing 4–manifolds, also uniquely describe open book decompositions.

It is well known that open books carry geometric information. Indeed, a theorem of Thurston-Winkelnkemper [38] shows that every open book decomposition supports a contact structure. A theorem of Honda-Kazez-Matic [27, 28] shows that the monodromy of an open book decomposition defines a contact invariant in the Heegaard Floer homology of the boundary 3–manifold. I am strongly interested in this topic because of the connection between 3–manifolds with an open book induced by a trisection, and contact manifolds that bound symplectic manifolds. The two important open problems in contact geometry in 3–dimensions concern the existence and classification of tight contact structures. The easiest way to prove a contact structure is tight is to find a symplectic filling. A theorem of Gromov-Eliashberg says that if (M, ξ) is a weakly symplectically semi-fillable contact structure then ξ is tight. So, establishing a relationship between trisection fillings and symplectic fillings has the potential of giving us new information about tightness.

Ozváth and Szabó's $\widehat{HF}$ groups arise as the homology groups of certain Lagrangian intersection Floer chain complexes. The generators are determined by intersections of curves in a Heegaard diagram (Σ, α, β) of Y, and the differential counts holomorphic representatives of Whitney disks between them. If (Y, ξ) is a contact manifold, there exists an element $c(\xi) \in \widehat{HF}(-Y)$ which is an isotopy invariant of ξ, vanishes if ξ is overtwisted, and is a primitive element if ξ is Stein fillable. Extending the definition of the differential, Ozváth and Szabó also define maps $f_{\alpha\beta\gamma}$ and $h_{\alpha\beta\gamma\delta}$ that count holomorphic representatives of Whitney triangles and rectangles in diagrams of Heegaard triples and quadruples respectively. From a relative trisection diagram $(\Sigma, \alpha, \beta, \gamma)$ one obtains a Heegaard 4-tuple that sees the monodromy. Since each pair of collections of curves gives a Heegaard diagram for $\#^k(S^2 \times S^1)$, we have that $\widehat{HF}(-,-) = \Lambda^* H^1(\#^k(S^2 \times S^1); \mathbb{Z}/2\mathbb{Z})$ and so its top-dimensional group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, and it therefore has an easily identifiable generator. This generator is represented by intersection points between curves. These properties are particular to trisections and I expect them to simplify computations of the Ozváth and Szabó contact invariant, and to produce interesting invariants of contact structures.

I am very motivated to find a relationship between trisections and the contact invariant in Heegaard Floer homology. This project is at an early stage but I expect to bring it to maturity in the upcoming years. The following are a few questions that guide my research on this topic:

- Is the contact invariant always in the image of $h_{\alpha\beta\gamma\alpha'}$? If not, what conditions on the geometry of W will guarantee that the contact invariant is in the image of $h_{\alpha\beta\gamma\alpha'}$?

- Is the map $h_{\alpha\beta\gamma\alpha'}$ completely determined by $f_{\alpha\beta\gamma}$?
- Is the image of the top-generators under $h_{\alpha\beta\gamma\alpha'}$ an invariant of the contact structure?

3.2. Trisections and Symplectic Structures

Symplectic geometry plays a significant role in 4–manifold theory. The work of Donaldson [8] and Gompf [19, 20] on Lefschetz pencils shows that symplectic 4–manifolds can be studied using the mapping class group of surfaces, a well understood object. A trisection of a closed 4–manifold gives a combinatorial description of 4–manifolds as diagrams of curves drawn on surfaces, and so in a way, also the mapping class group. The AIM SQuaRE project "Trisections, Knotted Surfaces, and Symplectic 4-Manifolds", joint with P. Lambert-Cole, J. Meier, P. Melvin and L. Starkston focuses on the study of trisections and bridge trisections and hopes to find connections to symplectic and complex geometry. One of the goals of the proposed SQuaRE is to establish a theory of generalized bridge trisections for symplectic surfaces and for immersed surfaces with nodal and cuspidal singularities. We hope to use these to describe important 4–dimensional surgery operations from the perspective of trisections, and to catalog the bridge trisections of important examples in 4–manifold topology. An additional goal is to specialize trisection theory to the class of symplectic 4-manifolds and produce new invariants of symplectic 4–manifolds using trisections. Finally, since symplectic 4–manifolds have been extensively studied in terms of both surgery constructions and branched covers, we will import the theory of relative trisections to effectively study the pieces of 4–manifolds involved in surgeries, and we will use the branched covers and surgery constructions to augment the collection of examples of trisections of symplectic 4–manifolds. This is a new project and we are very excited about the doors that this exploration will open.

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